TBA TYPE EQUATIONS AND TROPICAL CURVES

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Abstract. The Joyce integrable system and the corresponding Bridgeland-Toledano Laredo connections are fundamental objects associated with suitable abelian categories or, more generally, with a class of continuous families of stability data. We offer an overview of some of our work, mostly joint with M. Garcia Fernandez, focussing on equations of TBA type as a useful tool in the analysis of these objects and their deformations, and as a means to establish a connection with tropical geometry.

1. Introduction

The present paper has two main aims. Firstly we offer a brief and hopefully accessible overview of (part of) our rather long work [FGS] (joint with M. Garcia-Fernandez), motivated by certain aspects of the fundamental work of Joyce and Bridgeland-Toledano Laredo [J], [BT1] and of the equally important physical work of Gaiotto, Moore and Neitzke [GMN1]. In particular [FGS] studies the interactions between Kontsevich-Soibelman stability data, irregular meromorphic connections, and collapse to tropical geometry. This was the topic of the first author's talk at VBAC 2014. On the other hand we include a short discussion of a possible q-deformation of these results which appeared in the older draft [FS1] (of which we are borrowing the name) but was left out of [FGS]. This q-deformation appears to have some interest, especially since recently it has been rediscovered by physicists [CNV]. In a nutshell, our aim is to explain the curious-looking TBA ("Thermodynamic Bethe Ansatz") which appears in the title and what it has to do with Joyce's integrable system, the Bridgeland-Toledano Laredo connections and tropical geometry. The first two objects are reviewed in section 2. Our main results on the BTL connections and their deformations inspired by mathematical physics are presented in section 3. Finally section 4 sketches the collapse to tropical geometry and its q-deformation.

2. The Joyce integrable system

We take as our starting point an integrable system of partial differential equations introduced by D. Joyce [J]. To write it down we need a finite rank lattice Γ endowed with a skew-symmetric bilinear form $\langle -, - \rangle$.

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Definition 2.1. The Kontsevich-Soibelman (commutative, associative) algebra $\mathfrak{g}_\Gamma$ is the group algebra $\mathbb{C}[\Gamma]$ endowed with the Lie bracket induced by $\langle -,-\rangle$: it is generated by monomials $e_\alpha, \alpha \in \Gamma$ with bracket $[e_\alpha, e_\beta] = \langle \alpha, \beta\rangle e_{\alpha+\beta}$.

The Lie algebra $\mathfrak{g}_\Gamma$ is in fact Poisson, i.e. inner Lie algebra derivations act as commutative algebra derivations. Fixing a strictly convex cone in $\Gamma \otimes \mathbb{C}$ defines a monoid $\Gamma_{\geqslant 0} \subset \Gamma$ and a corresponding Poisson subalgebra $\mathfrak{g}_{\geqslant 0} \subset \mathfrak{g}_\Gamma$. Similarly there is a monoid $\Gamma_{>k} \subset \Gamma_{\geqslant 0}$ given by elements which can be written as the sum of more than k nonzero lattice points in $\Gamma_{\geqslant 0}$, and a corresponding ideal $\mathfrak{g}_{>k} \subset \mathfrak{g}_{\geqslant 0}$. We will always assume that $\Gamma_{>0}$ contains a basis for the lattice Γ.

Definition 2.2. The completed algebra $\widehat{\mathfrak{g}}_\Gamma$ (with respect to a fixed choice of cone as above) is the inverse limit of the inverse system of finite-dimensional nilpotent Lie algebras $\mathfrak{g}_{\leqslant k} = \mathfrak{g}_{\geqslant 0}/\mathfrak{g}_{>k}$.

In the following we write $\mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$ for the group of automorphisms of $\widehat{\mathfrak{g}}_\Gamma$ as an associative, commutative algebra (not necessarily preserving the Lie bracket). Similarly we write $D^*(\widehat{\mathfrak{g}}_\Gamma)$ for the Lie algebra of derivations of $\widehat{\mathfrak{g}}_\Gamma$ as a commutative algebra. We will often work in the Lie algebra extension of $\widehat{\mathfrak{g}}_\Gamma$ by the group $\mathrm{Hom}(\Gamma, \mathbb{C})$: for Z a linear map in the latter we set $[Z, e_\alpha] = Z(\alpha)e_\alpha$. Notice that this same expression embeds $\mathrm{Hom}(\Gamma, \mathbb{C})$ in $D^*(\widehat{\mathfrak{g}}_\Gamma)$, and as usual $\widehat{\mathfrak{g}}_\Gamma$ also embeds in $D^*(\widehat{\mathfrak{g}}_\Gamma)$ by inner derivations $[e_\alpha, -]$. We will write exp for the Lie algebra exponential, so for $g \in \widehat{\mathfrak{g}}_\Gamma$ the symbol $\exp(g)$ lies in the (inverse limit) Lie group $\exp(\widehat{\mathfrak{g}}_\Gamma)$. It will be important for us not to confuse this with the exponential of g as an element of a complete commutative associative algebra, so we write $\exp_*(g)$ for the latter.

Holomorphic functions with values in $\widehat{\mathfrak{g}}_\Gamma$ are given by inverse limits of holomorphic functions with values in the finite dimensional vector spaces $\mathfrak{g}_{\leqslant k}$. Similarly all our forthcoming statements about holomorphic functions with values in $\mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$, as well as meromorphic connections with values in $\widehat{\mathfrak{g}}_\Gamma$ or $D^*(\widehat{\mathfrak{g}}_\Gamma)$ and their monodromy are really statements about their projections to $\mathfrak{g}_{\leqslant k}$ for all k. See [BT2] for a more formal treatment.

We are especially interested in $\widehat{\mathfrak{g}}_\Gamma$-valued holomorphic functions $f(Z)$ of a variable Z which lies in an open subset of $\mathrm{Hom}(\Gamma, \mathbb{C})$. These complex vectors Z are usually called *central charges* in the mathematical physics literature. In particular we always assume that Z does not vanish on points of $\Gamma_{>0}$. We denote the graded components by $f^\alpha(Z)$ ($\alpha \in \Gamma_{\geqslant 0}$), so $f^\alpha(Z) = \tilde{f}^\alpha(Z)e_\alpha$ for a complex-valued holomorphic $\tilde{f}^\alpha(Z)$.

Definition 2.3. The Joyce system of PDEs is

$$df^\alpha(Z) = \sum_{\beta+\gamma=\alpha} [f^\beta(Z), f^\gamma(Z)] \frac{d(Z(\beta))}{Z(\beta)} \tag{2.1}$$

(here and below we sum over $\beta, \gamma \in \Gamma_{>0}$). Equivalently in terms of the collection of holomorphic functions $\tilde{f}^\alpha(Z)$ we have

$$d\tilde{f}^\alpha(Z) = \sum_{\beta+\gamma=\alpha} \langle \beta, \gamma\rangle \tilde{f}^\beta(Z)\tilde{f}^\gamma(Z) \frac{d(Z(\beta))}{Z(\beta)}. \tag{2.2}$$

Remark 2.4. The equations (2.2) are a a first order system of holomorphic integrable PDEs, so there is a unique solution to the Cauchy problem $f(Z_0) = f_0$ in an open neighbourhood of a generic $Z_0 \in \mathrm{Hom}(\Gamma, \mathbb{C})$.

Joyce realised that there is an *explicit* solution of (2.1), with special properties, attached to each finite-dimensional complex algebra A of homological dimension 1, e.g. the path algebra $\mathbb{C}Q$ of a finite quiver Q without oriented cycles.

Let $\mathcal{A}$ be the abelian category of finite-dimensional complex representations of A (finite dimensional complex A-modules). We take the lattice Γ to be the Grothendieck group $K(\mathcal{A}) \cong \mathbb{Z}^N$ (where N is the number of simple objects), and choose as $\langle -, - \rangle$ the skew-symmetrisation of the Euler form on $K(\mathcal{A})$. The effective classes define a cone $K_{>0}(\mathcal{A}) \subset K(\mathcal{A})$. The Joyce solution is defined for $Z \colon \Gamma \to \mathbb{C}$ varying in the locus of homomorphisms mapping $K_{>0}(\mathcal{A})$ to the upper half-plane $\mathbb{H}$, a domain biholomorphic to $\mathbb{H}^N$. Such a Z defines naturally a stability condition for A-modules in the sense of King [K] (simply by the taking the slope of an object of class α to be the principal argument of $Z(\alpha)$). In a long series of papers Joyce constructed "invariants" $J(\alpha, Z) \in \mathbb{Q}$ counting semistable objects of class α (see [J] for full references). Let $U\mathfrak{g}_\Gamma$ be the universal enveloping algebra of $\mathfrak{g}_\Gamma$. One can show that the product of monomials $e_{\alpha_1}, \dots, e_{\alpha_n}$ taken in $U\mathfrak{g}_\Gamma$ is a rational multiple of the commutative product $\prod_i e_{\alpha_i} = e_{\alpha_1 + \cdots + \alpha_n}$. We denote this rational weight by $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}$.

Theorem 2.5 (Joyce [J])**.** *There exist* sectionally holomorphic *functions J_n, holomorphic on a dense open subset of $\mathbb{H}^n$, such that the functions*

$$\widetilde{f}^{\alpha}(Z) = \sum_{n \geq 1} \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} J_n(Z(\alpha_1), \cdots, Z(\alpha_n)) \prod_i J(\alpha_i, Z)\, e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} \tag{2.3}$$

are in fact holomorphic *and give a solution to* (2.2)*. There are (complicated) explicit formulae for J_n and $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}$ as sums over oriented graphs. The functions J_n are* universal*: they do not depend on the particular choice of algebra A.*

Notice that we are following our convention of always summing over elements of $\Gamma_{>0}$, so the sum (2.3) contains only finitely many terms (there are only finitely many possible decompositions of α into classes in $K_{>0}(\mathcal{A})$). The $\widetilde{f}^{\alpha}(Z)$ are called *Joyce holomorphic generating functions* for the Joyce "invariants" counting semistable objects in class α. Bridgeland and Toledano Laredo [BT1] also established a remarkable explicit formula for J_n as a sum of multilogarithms attached to rooted oriented trees. Later on we will explain alternative formulae proved in [FGS]. Notice that the Joyce functions J_n and the resulting $f^{\alpha}(Z) = \widetilde{f}^{\alpha}(Z) e_{\alpha}$ are homogeneous under positive real scalings: for $R \in \mathbb{R}_{>0}$ we have

$$J_n(RZ(\alpha_1), \cdots, RZ(\alpha_n)) = J_n(Z(\alpha_1), \cdots, Z(\alpha_n)),$$
$$f^{\alpha}(RZ) = f^{\alpha}(Z).$$

In view of the connections with mathematical physics that will be explained shortly, this should be regarded as a conformal invariance (with $|Z|$ acting as an energy scale).

Remark 2.6. The existence of Joyce's explicit solution to (2.1) attached to the abelian category $\mathcal{A}$ is somewhat reminiscent of the special solutions of the Cecotti-Vafa tt^* integrable system [CV] attached to quantum cohomology and Landau-Ginzburg models. This analogy is still poorly understood. For example the physical work [GMN2] seems to suggest that there should exist a very interesting system coupling a deformation of the Joyce system (which we will describe in detail soon) and the tt^* equations.

2.1. **Isomonodromy perspective.** Bridgeland and Toledano Laredo [BT1] realised that (2.1) is precisely the isomonodromy condition for a family of meromorphic irregular connections on $\mathbb{P}^1$ of a special form. We are not able to offer a review of irregular connections and their generalised monodromy here, but refer the reader to [BT2] which also contains an extensive list of references.

Let $f(Z)$ be the $\widehat{\mathfrak{g}}_{\Gamma}$-valued holomorphic function defined by

$$f(Z)=\sum_{\alpha}\widetilde{f}^{\alpha}(Z)e_{\alpha}$$

with $f^{\alpha}(Z)$ as in Theorem 2.5.

Theorem 2.7 (Bridgeland-Toledano Laredo [BT1])**.** *The family of meromorphic connections on $\mathbb{P}^1$, with values in the extension of $\widehat{\mathfrak{g}}_{\Gamma}$ by* $\mathrm{Hom}(\Gamma,\mathbb{C})$*, given by*

$$\nabla^{BTL}(Z)=d-\left(\frac{Z}{t^2}+\frac{f(Z)}{t}\right)dt \tag{2.4}$$

has constant generalised monodromy.

Indeed the equations (2.1) on a general $f(Z)$ are precisely the condition that the family $d-\left(\frac{Z}{t^2}+\frac{f(Z)}{t}\right)dt$ be isomonodromic. We will not explain this notion in full detail, but recall that in the presence of a pole of order 2 one needs to extend the monodromy data of a connection of the form (2.4) by including the jumps of its canonical flat sections across certain distinguished (Stokes) rays: these are called Stokes factors. For $\nabla^{BTL}(Z)$ the Stokes factor along a ray $\ell(Z)$ takes the form

$$S_{\ell(Z)}=\exp(\xi_{\ell}(Z))$$

for a nonzero Lie algebra element $\xi_{\ell}(Z)$. Fixing a convex sector $V\subset\mathbb{H}$, the isomonodromy condition says that the slope-ordered product

$$\overrightarrow{\prod_{\ell(Z)\in V}}\exp(\xi_{\ell}(Z)) \tag{2.5}$$

remains constant as long as no Stokes ray $\ell(Z)$ crosses ∂V. One can show that indeed ξ_{ℓ} is the element of $\widehat{\mathfrak{g}}_{\Gamma}$ given by

$$\xi_{\ell}(Z)=\sum_{Z(\alpha)\in\ell}J(\alpha,Z)e_{\alpha}. \tag{2.6}$$

The theory developed in [J], [BT1] then shows that the crucial isomonodromic condition (2.5) is equivalent to the Harder-Narashiman property for $\mathcal{A}$ combined with the "motivic" properties of Joyce's invariants $J(\alpha, Z)$, in a way pionereed by Reineke [R].

2.2. **Stability data and Poisson automorphisms.** Kontsevich-Soibelman [KS] introduced the notion of a continuous family of stability data on a graded Lie algebra, which in the special case of $\widehat{\mathfrak{g}}_\Gamma$ axiomatises the factorisation property of the Joyce invariants $J(\alpha, Z)$ given by (2.5), (2.6). Fix a general symplectic lattice Γ with cone $\Gamma_{>0} \subset \Gamma$ and let $U \subset \mathrm{Hom}(\Gamma, \mathbb{C})$ be an open subset such that $Z(\Gamma_{>0}) \subset \mathbb{H}$ for all $Z \in U$.

Definition 2.8. A continuous family of stability data on $\widehat{\mathfrak{g}}_\Gamma$ parametrised by an open subset $U \subset \mathrm{Hom}(\Gamma, \mathbb{C})$ as above is a collection of algebra elements $\{\widetilde{a}(\alpha, Z)e_\alpha\}$, $\alpha \in \Gamma_{>0}$ with $\widetilde{a}(\alpha, Z) \in \mathbb{Q}$ and such that for each strictly convex positive cone $V \subset \mathbb{H}$ the group element

$$\overrightarrow{\prod_{\ell \subset V}} \exp\left(\sum_{Z(\alpha)\in \ell} \widetilde{a}(\alpha, Z)e_\alpha\right) \in \widehat{\mathfrak{g}}_\Gamma$$

is constant as long as the set of rays ℓ with nonvanishing $\sum_{Z(\alpha)\in\ell} \widetilde{a}(\alpha, Z)e_\alpha$ does not cross ∂V.

Thus when $\Gamma = K(\mathcal{A})$ the Joyce invariants "counting" semistable A-modules define a continuous family of stability data on $\widehat{\mathfrak{g}}_\Gamma$ parametrised by central charges with $Z(K_{>0}(\mathcal{A})) \subset \mathbb{H}$, that is by $\mathbb{H}^N$, given by

$$\widetilde{a}(\alpha, Z) = J(\alpha, Z).$$

Theorem 2.9 (Bridgeland-Toledano Laredo [BT1])**.** *Let $\{\widetilde{a}(\alpha, Z), \alpha \in \Gamma\}$ be a continuous family of stability data on $\widehat{\mathfrak{g}}_\Gamma$ parametrised by U. There exists a family of irregular meromorphic connections on $\mathbb{P}^1$, with values in the extension of $\widehat{\mathfrak{g}}_\Gamma$ by* $\mathrm{Hom}(\Gamma, \mathbb{C})$*, of the form*

$$\nabla^{BTL}(Z) = d - \left(\frac{Z}{t^2} + \frac{v(Z)}{t}\right) dt$$

with Stokes factors

$$S_\ell(Z) = \exp\left(\sum_{Z(\alpha)\in\ell} \widetilde{a}(\alpha, Z)e_\alpha\right). \tag{2.7}$$

The residue $v(Z) = \sum_\alpha v^\alpha(Z)e_\alpha \in \widehat{\mathfrak{g}}_\Gamma$ is given explicitly by

$$v^\alpha(Z) = \sum_{n\geq 1} \sum_{\alpha_1+\cdots+\alpha_n=\alpha} J_n(Z(\alpha_1), \cdots, Z(\alpha_n)) \prod_i \widetilde{a}(\alpha_i, Z)\, e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n},$$

and for $R \in \mathbb{R}_{>0}$ one has $v(RZ) = v(Z)$.

Bridgeland and Toledano Laredo's proof uses a combination of the Fourier-Laplace transform and an explicit inversion formula for non-commutative formal power series.

Globally the connection $\nabla^{BTL}(Z)$ has the "conformal invariance" property of being unchanged under the combined (positive real) scalings $Z \mapsto RZ, t \mapsto Rt$:

$$\nabla(Z) \mapsto d - \left(\frac{RZ}{R^2t^2} + \frac{v(RZ)}{Rt}\right) Rdt = \nabla(Z).$$

It will be important for us to regard the Stokes factors (2.7) as products of explicit "symplectomorphisms".

Definition 2.10. For a general lattice Γ we say that a central charge $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ is generic if elements e_α, e_β with $Z(\alpha), Z(\beta)$ lying on the same ray ℓ have vanishing Lie bracket (i.e. $\langle \alpha, \beta \rangle = 0$). We say that Z is strongly generic if $Z(\alpha), Z(\beta)$ lying on the same ray ℓ implies that α, β are linearly dependent. We write $\mathrm{Hom}^0(\Gamma, \mathbb{C})$ for the locus of strongly generic central charges.

For $\Omega \in \mathbb{Q}$ let T^{Ω}_{β} denote the element of $\mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$ given by

$$T^{\Omega}_{\beta}(e_\alpha) = e_\alpha (1 - e_\beta)^{\Omega \langle \beta, \alpha \rangle}.$$

One can prove that in fact T^{Ω}_{β} is a Poisson automorphism (it preserves the Lie bracket). Kontsevich-Soibelman noticed that for generic Z there is a factorisation in $\mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$

$$\exp_{D^*(\widehat{\mathfrak{g}}_\Gamma)} \left(\sum_{Z(\alpha) \in \ell} \widetilde{a}(\alpha, Z) [e_\alpha, -] \right) = \prod_{Z(\beta) \in \ell} T^{\Omega(\beta, Z)}_{\beta}, \tag{2.8}$$

where $\Omega(\beta, Z)$ is given by the Möbius transform

$$\widetilde{a}(\alpha, Z) = \sum_{\alpha = k\beta} \frac{1}{k^2} \Omega(\beta, Z).$$

The continuity condition becomes the constraint that the product of Poisson automorphisms $\prod^{\to}_{\ell \subset V} \prod_{Z(\alpha) \in \ell} T^{\Omega(\alpha, Z)}_{\alpha}$ remains constant in the locus of generic central charges (even when crossing the nongeneric locus) as long as no rays supporting a nonvanishing factor enter or leave V.

3. Deformations of ∇^{BTL} inspired by mathematical physics

Ideas very close to those of the previous section also appeared independently in the physical work of Gaiotto, Moore and Neitzke [GMN1]. This work is important for us since it suggests a natural class of deformations of the Joyce system and of BTL connections.

According to [GMN1] many important aspects of the physics of a class of $\mathcal{N} = 2$ supersymmetric field theories in four dimensions can be captured by a family of connections on $\mathbb{P}^1$ of the form

$$\nabla^{GMN}(Z) = d - \left(\frac{\mathcal{A}^{(-1)}(Z)}{z^2} + \frac{\mathcal{A}^{(0)}(Z)}{z} + \mathcal{A}^{(1)}(Z) \right) dz$$

with values in the Lie algebra of complex-valued smooth vector fields on a real torus $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}$. Physical constraints on Z imply that the family is actually parametrised by a half-dimensional submanifold $\mathcal{B} \subset \mathrm{Hom}(\Gamma, \mathbb{C})$. Crucial quantities of the theory can be computed from flat sections of $\nabla^{GMN}(Z)$. This applies perhaps most importantly to the (hyperKähler) metric on the moduli space $\mathcal{M}$ of the theory compactified on S^1. In this case the coefficients $\mathcal{A}^{(i)}(Z)$ are smooth, real-analytic functions of Z, but the basic requirement that one is working with continuous deformations of a fixed theory still imposes the condition that the connections $\nabla^{GMN}(Z)$ have constant generalised monodromy, given by "symplectomorphisms" of the complex algebraic torus $\Gamma \otimes_{\mathbb{Z}} \mathbb{C}^*$, which have precisely the form (2.8). The underlying collection of numbers $\{\widetilde{a}(\alpha, Z), \alpha \in \Gamma\}$ are (the Möbius transform of) the natural physical "indices" attached to BPS states of the theory.

Remark 3.1. Recently Kontsevich [Ko] proposed a general picture of infinite-dimensional isomonodromy problems in complex and symplectic geometry, in which the GMN hyperKähler construction appears as a special case.

Constructing the connections $\nabla^{GMN}(Z)$ mathematically seems a rather daunting task, and as far as we know the problem is still open (although recently there has been some progress, see for example [IN]). One of our motivations for writing [FGS] was to give a very rough but rigorous analogue of $\nabla^{GMN}(Z)$ in a purely algebraic setting, in particular making it possible to compare with the Bridgeland-Toledano Laredo connections (2.4). However it turns out that the resulting purely algebraic GMN connections have other interesting properties. Perhaps the most important feature is that they allow to make contact with the geometry of rational tropical curves immersed in the plane and the Gross-Pandharipande-Siebert approach to wall-crossing based on tropical enumerative invariants [GPS]. Let us start with a basic existence result.

Theorem 3.2 ([FGS] section 4.1)**.** *Let $\{\widetilde{a}(\alpha, Z), \alpha \in \Gamma\}$ be a continuous family of stability data on $\widehat{\mathfrak{g}}_\Gamma$ parametrised by U. There exists a family of irregular meromorphic connections on $\mathbb{P}^1$, with values in $D^*(\widehat{\mathfrak{g}}_\Gamma)$, of the form*

$$\nabla(Z) = d - \left(\frac{\mathcal{A}^{(-1)}(Z)}{z^2} + \frac{\mathcal{A}^{(0)}(Z)}{z} + \mathcal{A}^{(1)}(Z)\right) dz \tag{3.1}$$

with formal type at $z = 0$ given by

$$d - \frac{Z}{z^2} dz,$$

Stokes factors at $z = 0$

$$S_\ell(Z) = \prod_{Z(\alpha) \in \ell} T_\alpha^{\Omega(\alpha, Z)} \in \mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma),$$

and complex conjugate generalized monodromy at $z = \infty$ (where the Poisson automorphisms $T_\alpha^{\Omega(\alpha,Z)}$ are regarded as real). The family $\nabla(Z)$ has constant generalised monodromy (globally as a family of connections on $\mathbb{P}^1$).

The coefficients of (3.1) are constant derivations: we have $\mathcal{A}^{(i)}(Z) \in D^*(\widehat{\mathfrak{g}}_\Gamma)$, $\partial_z \mathcal{A}^{(i)}(Z) = 0$.

Remark 3.3. Notice that Stokes rays for $\nabla(Z)$ lie in the upper half-plane $\mathbb{H}$. This is the opposite of the physical convention (which is followed in [FGS]), but we decided not to change this sign in the present paper. This makes it easier to compare with the BTL connections, but leads to relative minus signs with respect to [FGS] and the mathematical physics literature.

The Fourier-Laplace transform approach does not apply to a $D^*(\widehat{\mathfrak{g}}_\Gamma)$-valued connection of the form (3.1) (because of the presence of double poles at $z=0$ and $z=\infty$). Instead our proof of Theorem 3.2 is based on the direct approach through integral equations proposed by physicists [GMN1]: this leads to the TBA type equation and a new set of explicit formulae. We treat these aspects below in section 3.1.

Our second basic result relates the "conformal limit" of $\nabla(Z)$ to the Bridgeland-Toledano Laredo connections. Unlike $\nabla^{BTL}(Z)$, the connection $\nabla(Z)$ is not "conformally invariant", but is mapped by $Z \mapsto RZ$, $z \mapsto Rz$ to

$$\nabla(Z) \mapsto d - \left(\frac{\mathcal{A}^{(-1)}(RZ)}{Rz^2} + \frac{\mathcal{A}^{(0)}(RZ)}{z} + R\mathcal{A}^{(1)}(RZ)\right) dz.$$

The derivations $\mathcal{A}^{(i)}(Z)$ are *not* homogeneous in Z, but become very complicated functions of R. This lack of conformal invariance makes perfect sense from a physical point of view: $\nabla(Z)$ is attached to an effective quantum field theory that does depend on an energy scale parameter, roughly $|Z|$. However one can easily check that the formal types of $\nabla(Z)$ at $z=0$ and $z=\infty$ have well definite finite limits under the scaling: in fact the first is invariant, while the latter vanishes as $R \to 0$. The upshot is that one may expect that the Bridgeland-Toledano Laredo connections $\nabla^{BTL}(Z)$ arise as the "conformal limit" of $\nabla(Z)$ as $R \to 0$:

$$\nabla^{BTL}(Z) \stackrel{?}{=} \lim_{R\to 0} \nabla(RZ)_{z=Rt}.$$

The existence of a conformal limit $\lim_{R\to 0}\nabla^{GMN}(RZ)_{z=Rt}$ for the full physical connections was hinted at in [GMN1]. We found in [FGS] that this cannot be true verbatim, at least in the algebraic, perturbative setting of $\nabla(Z)$: the formal type at $z=0$ receives corrections in $\nabla(RZ)_{z=Rt}$ which are *divergent* (logarithmically or worse) as $R \to 0$,

$$\nabla(RZ)_{z=Rt} \sim \nabla^{BTL}(Z) + \log(R)\mathcal{A}^{\log} dt + \log^2(R)\mathcal{A}^{\log^2} dt + \cdots$$

Our next result says that these divergencies can be gauged away with constant gauge transformations.

Theorem 3.4 ([FGS] Theorem 2.1)**.** *Fix a continuous family of stability data $\{\widetilde{a}(\alpha, Z), \alpha \in \Gamma\}$ on $\widehat{\mathfrak{g}}_\Gamma$ parametrised by U. There exists a 1-parameter family of constant gauge transformations $g(R) \in \mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$ such that*

$$\nabla^{BTL}(Z) = \lim_{R\to 0} g(R) \cdot \nabla(RZ)_{z=Rt}$$

(where $\nabla(Z)$ is given by Theorem 3.2 and $\nabla^{BTL}(Z)$ by Theorem 2.9).

The action of $g \in \mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$ is of course simply $d - \mathcal{A}\,dz \mapsto d - g\mathcal{A}g^{-1}dz$ (for all $\mathcal{A} \in D^*(\widehat{\mathfrak{g}}_\Gamma)$).

3.1. **TBA type equation.** As we mentioned a key aspect of the proof of Theorem 3.2 is that we follow very closely the original physical argument. In particular through Theorem 3.4 we also get a new proof of the result of Bridgeland and Toledano Laredo, Theorem 2.9. The connection $\nabla(Z)$ is reconstructed starting from a distinguished set of flat sections on (prospective) Stokes sectors $\Sigma \subset \mathbb{C}^*$ relative to $z = 0$. The complex conjugate generalised monodromy at $z = \infty$ follows from the construction. The reference for all results in this section is [FGS] section 4.

Let us fix a continuous family of stability data $\{\widetilde{a}(\alpha, Z), \alpha \in \Gamma\}$ on $\widehat{\mathfrak{g}}_\Gamma$ parametrised by U. A (prospective) "Stokes sector" Σ is an inverse limit under inclusion of sectors between consecutive rays ℓ with $S_\ell \neq 1$ in $\mathfrak{g}_{\leqslant k}$ (where S_ℓ is attached to the continuous family by the usual expression (2.8)). For fixed $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ consider an integral transform Λ_α, $\alpha \in \Gamma$, acting on suitable $\widehat{\mathfrak{g}}_\Gamma$-valued holomorphic functions $\phi(z)$ by

$$\Lambda_\alpha[\phi](z) = \int_{\ell_\alpha(Z)} \frac{dw}{w}\frac{w+z}{w-z}\log(1-\phi(w)).$$

The right-hand side makes sense by completeness of $\widehat{\mathfrak{g}}_\Gamma$ as long as the Γ-graded components of ϕ restrict to smooth functions in a neighbourhood of $\ell_\alpha(Z)$ which are in L^1 with respect to the measure $w^{-1}(w-z)^{-1}(w+z)dw$. The specific choice of integration kernel $w^{-1}(w-z)^{-1}(w+z)$ will ensure that $\nabla(Z)$ has double poles at $0, \infty$ with complex conjugate monodromy. Let $\phi(z)$ be a holomorphic function on a sector Σ with values in $\mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$. We define a new $\mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$-valued holomorphic function by

$$\mathcal{Z}[\phi](z)(e_\alpha) = e_\alpha \exp_*\left(-z^{-1}Z(\alpha) - z\bar{Z}(\alpha) + \sum_\gamma \Omega(\gamma, Z)\langle\gamma, \alpha\rangle\Lambda_\gamma[\phi(e_\gamma)](z)\right) \tag{3.2}$$

provided the components $\phi(z)(e_\gamma)$ satisfy the above regularity and integrability conditions.

Remark 3.5. The operator $\mathcal{Z}$ given by (3.2) is called a "Thermodynamic Bethe" type operator in [GMN1] by analogy with a similar construction in two-dimensional conformal field theory.

Define a holomorphic function $\mathbb{C}^* \to \mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$ by

$$X^0(z)(e_\alpha) = e_\alpha \exp_*(-z^{-1}Z(\alpha) - z\bar{Z}(\alpha)).$$

One checks that the new functions $X^{(i)}(z)$ given inductively by

$$X^{(i+1)}(z) = \mathcal{Z}[X^{(i)}](z)$$

are well defined and holomorphic $\mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$-valued functions in sectors Σ.

Proposition 3.6. *The sequence of functions $X^i(z)$ converges as $i \to \infty$ to a sectionally holomorphic function $X(z)\colon \mathbb{C}^* \to \operatorname{Aut}(\widehat{\mathfrak{g}}_\Gamma)$, holomorphic in sectors Σ. The limit $X(z)$ is a solution to the "Thermodinamic Bethe Ansazt" (TBA) type equation*

$$X(z) = \mathcal{Z}[X](z). \tag{3.3}$$

This means precisely that for all k the function $X(z)$ induces functions with values in $\operatorname{Aut}(\mathfrak{g}_{\leqslant k})$ which are well defined and holomorphic on the complement of the finitely many rays with $S_\ell \neq 1 \in \operatorname{Aut}(\mathfrak{g}_{\leqslant k})$, and satisfy (3.3) in this open region.

Proposition 3.7. *The condition* (3.3) *implies that $X(z)$ is sectionally holomorphic along a ray $\ell \subset \mathbb{C}^*$ with $S_\ell \neq 1$. The two sectional limits differ precisely by the Poisson automorphism S_ℓ.*

This is again a statement about the projections of $X(z)$ to $\operatorname{Aut}(\mathfrak{g}_{\leqslant k})$ for all k (where the number of rays with nonvanishing S_ℓ becomes finite).

Finally we can define a connection $\nabla(Z)$ on each sector Σ by

$$d - \mathcal{A}\,dz = d - \partial_z X(z) \circ X^{-1} dz \tag{3.4}$$

(again read this projecting to $\mathfrak{g}_{\leqslant k}$ for any k). One checks that the various connections on sectors glue to a resulting connection $\nabla(Z)$ which has the properties of Theorem 3.2.

3.2. **Explicit formulae.** The approach based on the TBA equation (3.3) leads to explicit formulae for flat sections $X(z)$ and the connection 1-form $\mathcal{A}\,dz$. Through Theorem 3.4 this yields a new set of formulae even in the case of the Bridgeland-Toledano Laredo connections (and Joyce's holomorphic generating functions). The results in this section are treated in [FGS] sections 4 and 6.

In the following we denote by T a rooted, connected tree whose vertices $v \in T$ are decorated by elements $\alpha(v) \in \Gamma_{>0}$. We denote the root decorations by α_T. The operation of removing the root produces a finite number of new connected, $\Gamma_{>0}$-decorated rooted trees $T \mapsto \{T_j\}$. Attached to T is a sectionally holomorphic function $G_T\colon \mathbb{C}^* \times \operatorname{Hom}^0(\Gamma, \mathbb{C}) \to \mathbb{C}^*$, uniquely determined by the recursion

$$G_T(z, Z) = \frac{1}{2\pi i} \int_{\ell_{\alpha_T}} \frac{dw}{w} \frac{w+z}{w-z} \exp(-Z(\alpha_T)w^{-1} - \bar{Z}(\alpha_T)w) \prod_j G_{T_j}(w, Z), \tag{3.5}$$

following the convention that an empty product equals 1.

Let us define weights $W_T(Z) \in \Gamma \otimes \mathbb{Q}$ attached to trees by

$$W_T(Z) = \frac{\widetilde{a}(\alpha_T, Z)}{|\operatorname{Aut}(T)|} \alpha_T \prod_{\{v \to w\} \subset T} \langle \alpha(v), \alpha(w) \widetilde{a}(\alpha(q), Z) \rangle.$$

Proposition 3.8. *We have $X(z) = Y(z)X^0(z)$, where*

$$Y(z, Z)(e_\beta) = e_\beta \exp \sum_T \prod_{v \in T} e_{\alpha(v)} \langle \beta, W_T(Z) \rangle G_T(z, Z). \tag{3.6}$$

It turns out that the automorphism $X(z)$ can be inverted explicitly thanks to (3.6) and the classical Lagrange inversion formula. Through (3.4) this yields an explicit formula for the connection 1-form $\mathcal{A}\,dz$. To state this we choose a basis γ_i, $i = 1, \dots, m$ for Γ contained in $\Gamma_{>0}$ and we introduce formal power series in m variables $s_1, \dots, s_m$ given by

$$\Phi_i(\mathbf{s}) = -z^{-1}Z(\gamma_i) - z\bar{Z}(\gamma_i) - \sum_T \prod_{v\in T} s^{\alpha(v)}\langle\gamma_i, W_T(Z)\rangle G_T(z, Z) \in \mathbb{C}[[\mathbf{s}]],$$

where for $\alpha = \sum_i \alpha_i\gamma_i$ we set $\mathbf{s}^\alpha = \prod_{i=1}^m s_i^{\alpha_i} \in \mathbb{C}[\mathbf{s}]$.

Proposition 3.9. *Let $\mathcal{A}\,dz$ be the connection 1-form of $\nabla(Z)$. Then we have*

$$\mathcal{A}(e_{\gamma_i}) = -\sum_{\alpha\in\Gamma_{>0}} [\mathbf{s}^\alpha]\det(\delta_{pq} + s_i\partial_q\Phi_p(\mathbf{s}))s_i \exp\left(\sum_j \alpha_j\Phi_j(\mathbf{s})\right)$$
$$X(e_\alpha)\sum_T \prod_{v\in T} e_{\alpha(v)}\langle\alpha, W_T(Z)\rangle\partial_z G_T(z, Z), \tag{3.7}$$

where the $[\mathbf{s}^\alpha]$ operator picks out the coefficient of the monomial s^α.

Remark 3.10. It is not clear a priori from this expression that $\mathcal{A}$ has the simple z-dependence given by (3.1), although it seems likely that this can be proved directly by induction on the length of T.

It is shown in [FGS] section 6 that the explicit formulae (3.6), (3.7) specialise through Theorem 3.4 to analogous results for $\nabla^{BTL}(Z)$. We define new weight functions $H_T(z, Z)$ by replacing the recursion (3.5) with

$$H_T(t, Z) = \frac{1}{2\pi i}\int_{\ell_{\alpha_T}} \frac{dw}{w}\frac{t}{w-t}\exp(-Z(\alpha_T)w^{-1})\prod_j H_{T_j}(w, Z),$$

The rational weights $W_T(Z)$ are unchanged.

Corollary 3.11. *The canonical flat sections of $\nabla^{BTL}(Z)$ are induced by the* $\mathrm{Aut}(\widehat{\mathfrak{g}}_\Gamma)$*-valued holomorphic function*

$$\widetilde{Y}(t, Z)(e_\beta) = e_\beta \exp\sum_T\prod_{v\in T} e_{\alpha(v)}\langle\beta, W_T(Z)\rangle H_T(t, Z).$$

Let $\widetilde{\mathcal{A}}(Z)\,dt$ be the connection 1-form of $\nabla^{BTL}(Z)$. Introduce formal power series in m variables $s_1, \dots, s_m$ given by

$$\Psi_i(\mathbf{s}) = -t^{-1}Z(\gamma_i) - \sum_T\prod_{v\in T} s^{\alpha(v)}\langle\gamma_i, W_T(Z)\rangle H_T(t, Z) \in \mathbb{C}[[\mathbf{s}]],$$

Then $\widetilde{\mathcal{A}}(Z)$ is the derivation acting by

$$\widetilde{\mathcal{A}}(Z)(e_{\gamma_i}) = -\sum_{\alpha\in\Gamma_{>0}} [\mathbf{s}^\alpha]\det(\delta_{pq} + s_i\partial_q\Psi_p(\mathbf{s}))s_i \exp\left(\sum_j \alpha_j\Psi_j(\mathbf{s})\right)$$
$$\widetilde{Y}(e_\alpha)\sum_T\prod_{v\in T} e_{\alpha(v)}\langle\alpha, W_T(Z)\rangle\partial_t H_T(t, Z). \tag{3.8}$$

Using (3.8) it is possible to reconstruct the inner derivation $[f(Z), -]$ as the coefficient of t^{-1} in the action of $\widetilde{\mathcal{A}}$.

Example 3.12. According to (3.8) there is a contribution to $\widetilde{\mathcal{A}}(Z)(e_{\gamma_i})$ corresponding to the choice $\alpha = \gamma_i$ given by

$$\widetilde{Y}(e_{\gamma_i}) \sum_T \prod_{v \in T} e_{\alpha(v)} \langle \gamma_i, W_T(Z) \rangle \partial_t H_T(t, Z).$$

Let us evaluate its contribution to (the inner derivation corresponding to) the Joyce function $[f(Z), -]$, in the approximation where $\widetilde{Y} \sim I$ and we only sum over single-vertex graphs.

Set $\rho(t, w) = (2\pi i)^{-1} t(w - t)^{-1}$, and notice that $\partial_t \rho = \frac{w}{t} \partial_w \rho$. Let us write $H_\alpha(t, Z)$ for the single single-vertex function attached to a decoration $\alpha \in \Gamma_{>0}$. Integrating by parts we find

$$\begin{aligned}
\partial_t H_\alpha(t, Z) &= \int_{\ell_\alpha(Z)} \frac{dw}{w} \partial_t \rho(t, w) \exp(-Z(\alpha) w^{-1}) \\
&= -\frac{1}{t} \int_{\ell_\alpha(Z)} dw \rho(t, w)\, \partial_w \exp(-Z(\alpha) w^{-1}) \\
&= -\frac{Z(\alpha)}{t} \int_{\ell_\alpha(Z)} \frac{dw}{w^2} \rho(t, w)\, \exp(-Z(\alpha) w^{-1}) \\
&= \frac{1}{2\pi i} \frac{1}{t} + O(t^2).
\end{aligned}$$

So in the current approximation we find

$$\widetilde{\mathcal{A}}(Z)(e_{\gamma_i}) \sim \frac{1}{2\pi i} \sum_\alpha \langle \gamma_i, W_\alpha(Z) \rangle e_{\alpha + \gamma_i}.$$

In other words as in [J] we have to a first order approximation

$$\tilde{f}^\alpha(Z) \sim \frac{1}{2\pi i} W_\alpha(Z).$$

4. TBA and tropical curves

Theorem 3.4 constructs $\nabla^{BTL}(Z)$ as a scaling limit of $\nabla(Z)$. As a consequence the behaviour of $\nabla^{BTL}(Z)$ under rescaling the central charge $Z \mapsto RZ$ is straightforward: only the principal part changes $t^{-2} Z \mapsto R t^{-2} Z$, while the residue is fixed. The situation for $\nabla(Z)$ is very different: the residue of $\nabla(RZ)$ (say at $z = 0$) is a complicated function of R, and its $R \to \infty$ behaviour becomes much more interesting. We will study this large R behaviour through the corresponding behaviour of solutions $X(Z)$ to the TBA type equation (3.3) given by (3.6) as Z crosses the critical locus of non-generic central charges. This behaviour turns out to be closely related to the enumerative geometry of rational tropical curves immersed in $\mathbb{R}^2$ (see [FGS] section 2.11 for a discussion of why such a connection may be expected from a mirror-symmetric viewpoint). We cannot recall the necessary background from tropical geometry here, but refer the reader to [FGS] section 7 for an exposition of the few notions which are actually needed.

Our aim is to study in detail the model case when $\mathcal{A}$ is the category of representations of the κ-Kronecker quiver Q_κ. Then $\mathcal{A} = \mathbb{C}Q_\kappa$-mod is a finite length abelian category of homological dimension 1, with Grothendieck group $K(\mathcal{A}) \cong \mathbb{Z}^2$ generated by the classes of the two simple objects $S_1 = (0 \to 1)$, $S_2 = (1 \to 0)$. The space of stability conditions $\mathbb{H}^2$ contains an open chamber $\mathcal{C}^+ \subset \mathbb{H}^2$ such that for $Z \in \mathcal{C}^+$ the only Z-stable objects are the simple representations S_i, $i = 1, 2$. The chamber $\mathcal{C}^+$ is cut out by the condition that the phase of $Z([S_1])$ in the upper half-plane is strictly larger than the phase of $Z([S_2])$. $\mathcal{C}^+$ is called the strong coupling region in mathematical physics. The open subset $\mathcal{C}^- = \mathbb{H}^2 \setminus \overline{\mathcal{C}^+}$ is the weak coupling region, and $\mathcal{W} = \partial\mathcal{C}^+ = \partial\mathcal{C}^-$ is called the wall of marginal stability.

We can now work with the Joyce continuous family of stability data on $\widehat{\mathfrak{g}}_\Gamma$ attached to the quiver Q_κ, parametrised by $\mathbb{H}^2$. In particular for $Z \in \mathcal{C}^+$ the function $\Omega(-, Z)$ on Γ vanishes except for $\Omega([S_i], Z) = 1$, $i = 1, 2$. We can construct the relevant family of connections $\nabla(Z)$ parametrised by $\mathrm{Stab}(Q_\kappa) = \mathbb{H}^2$ as in section 3.1. Fix a reference point $z^* \in -\mathbb{H}$. In the following we assume that all functions of $z \in \mathbb{C}^* \subset \mathbb{P}^1$ are in fact evaluated at $z = z^*$. Let $R \in \mathbb{R}_{>0}$ denote a positive real scaling parameter. The large R behaviour of the "building blocks" $G_T(RZ)$ of $X(RZ)$ across $\partial\mathcal{C}^+$ is governed by the following result.

Theorem 4.1 ([FGS] Theorem 2.2)**.** *Let $Z^\pm \in \mathcal{C}^\pm$, and let $G_T(Z^+)$ denote an n-vertex function appearing in the expansion* (3.6) *for flat sections of $\nabla(Z)$. There exists an expansion*

$$G_T(RZ^+) = \sum_{T'} \pm G_{T'}(RZ^-) + r(|Z^+ - Z^-|, R) \tag{4.1}$$

where $r(|Z^+ - Z^-|) \to 0$ as $|Z^+ - Z^-| \to 0$, and we sum over a finite set of rooted decorated trees T', not necessarily distinct. Let $\beta = \sum_{v \in T} \alpha(v)$.

The terms corresponding to a single-vertex tree in (4.1) *are labelled by a finite set of graphs C_i containing $n+1$ external* 1*-valent vertices and with* 3*-valent internal vertices. These terms are all equal to $G_\beta(Z^-, R)$ up to sign, and differ by a well defined factor $\varepsilon(C_i) = \pm 1$ which is uniquely attached to the graph C_i.*

Moreover the graphs C_i come naturally with extra combinatorial data which endows them with the structure of the combinatorial types of a finite set of rational tropical curves *immersed in the plane $\mathbb{R}^2$.*

Finally the single-vertex terms in (4.1) *are uniquely characterised by their asymptotic behaviour: they are of order*

$$(2|Z^-(\beta)|R)^{-1} \exp(-2|Z^-(\beta)|R)$$

as $R \to \infty$, uniformly *as $Z^- \to \mathcal{W}$.*

Example 4.2. The simplest nontrivial example of (4.1) is the expansion

$$G_{[S_1]\to[S_2]}(RZ^+) = G_{[S_1]+[S_2]}(RZ^-) + G_{[S_1]\to[S_2]}(RZ^-) + r(|Z^+ - Z^-|, R)$$

which follows from the standard residue theorem of complex analysis. The single-vertex term $G_{[S_1]+[S_2]}(RZ^-)$ is characterised by its *uniform* (in Z^-) $R \to \infty$ asymptotic behaviour

$$(2|Z^-([S_1] + [S_2])|R)^{-1} \exp(-2|Z^-([S_1] + [S_2])|R).$$

The graph C attached to this single-vertex term has a single 3-valent vertex attached to 3 distinct legs. It is identified naturally with the combinatorial type of a rational tropical curve immersed in $\mathbb{R}^2$: the balancing condition at the single 3-valent vertex reads simply

$$-[S_1] - [S_2] + [S_1 + S_2] = 0.$$

Theorem 4.1 says precisely that this simple-minded analysis can be carried out in a similar way for all n-vertex functions $G_T(Z^+)$, yielding a finite set of tropical types C_i. The tropical balancing condition at a 3-valent vertex of C_i with incident edges decorated by $\alpha_i \in \Gamma$ always takes the form $-\alpha_1 - \alpha_2 + \alpha_3 = 0$, and arises naturally from the residue theorem and the linearity of the central charge Z.

Theorem 4.1 establishes the large R tropical behaviour of the building blocks $G_T(RZ)$ of $X(RZ)$. We can then plug the expansion (4.1) into the expression (3.6) for $X(Z)$ and use the continuity of $X(Z)$ across $\partial\mathcal{C}^+$ to obtain more refined information on (weighted) sums of the contributions $\varepsilon(C_i) = \pm 1$ over tropical types C_i. To state a precise result we first notice that the n external, incoming 1-valent vertices of a graph C_i are necessarily labelled by a collection of positive integer multiples of $[S_1]$ and $[S_2]$, which we denote by $(w_{1i}[S_1], w_{2j}[S_2])$. In the light of Theorem 4.1 it is natural to regard this collection as defining a tropical degree $\mathbf{w} = (w_{1i}, w_{2j})$ for plane tropical curves. The weight vector $\mathbf{w}$ attached to a graph C_i depends only on the decorated tree T from which it originates. We denote by $C_i(T)$ the finite set of tropical types attached to T by Theorem 4.1. The enumerative invariant of rational tropical curves in $\mathbb{R}^2$ of degree $\mathbf{w}$ is denoted by $N^{\mathrm{trop}}(\mathbf{w}) \in \mathbb{N}$. A precise connection with Gromov-Witten theory through relative invariants counting rational curves with tangency conditions is given in [GPS] sections 3 and 4.

Theorem 4.3 ([FGS] Theorem 2.3)**.** *The sum of contributions $\varepsilon(C_i(T)) = \pm 1$ over tropical types C_i, weighted by the coefficients W_T in the expansion* (3.6) *for flat sections in $\mathcal{C}^+$,*

$$\sum_{\deg(T)=\mathbf{w}} W_T(Z^+) \sum_i \varepsilon(C_i(T))$$

equals the tropical invariant $N^{\mathrm{trop}}(\mathbf{w})$ enumerating plane rational tropical curves, times the combinatorial factor in $\Gamma \otimes \mathbb{Q}$ given by

$$\kappa^{l_1+l_2} \frac{1}{|\operatorname{Aut}(\mathbf{w})|} \prod_{k,l} \frac{1}{w^2_{kl}} (|\mathbf{w}|_1[S] + |\mathbf{w}|_2[T]).$$

Our proof uses the beautiful theory of the tropical vertex group developed in [GPS].

Example 4.4. We illustrate the result when $\kappa = 1$, $\mathbf{w} = (1+1, 1+2)$. The relevant diagrams and their total contributions $\sum_i \varepsilon(C_i(T))$ are given by

$$T_1 = \{[S_1] \to [S_2] \to [S_1] \to 2[S_2]\} \sim 0, \quad T_2 = \{[S_1] \leftarrow [S_2] \leftarrow [S_1] \to 2[S_2]\} \sim 0$$
$$T_3 = \{[S_1] \to 2[S_2] \to [S_1] \to [S_2]\} \sim 1, \quad T_4 = \{[S_1] \leftarrow 2[S_2] \leftarrow [S_1] \to [S_2]\} \sim 1$$

Computing with Theorem 4.1 gives

$$N^{\mathrm{trop}}(1+1, 1+2) = 8.$$

More precisely, let C_1, C_2 be the two distinct tropical types appearing in Figure 1. We write C_1 for the smooth type and C_2 for the nodal one. The tropical types

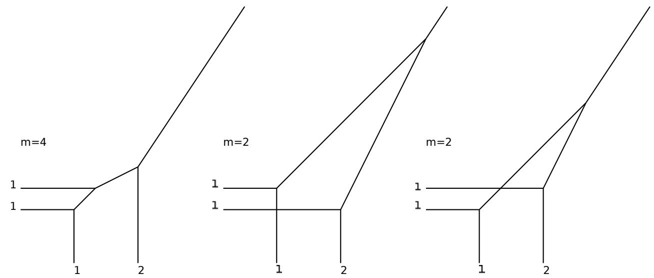

FIGURE 1. $N^{\mathrm{trop}}(1+1, 1+2) = 8$

appearing in Theorem 4.1 (with corresponding weights) are

$$\prod w_{ij}^2 W_{T_1} T_1 \sim \prod w_{ij}^2 W_{T_2} T_2 \sim -2C_1 + 2C_2,$$
$$\prod w_{ij}^2 W_{T_3} T_3 \sim \prod w_{ij}^2 W_{T_4} T_4 \sim 4C_1$$

so the total cycle of tropical types over T_i is $4C_1 + 4C_2$. On the other hand we can compute $N^{\mathrm{trop}}(1+1, 1+2) = 8$ directly by choosing (weighted) boundary conditions as shown in Figure 1. In this example the two computations agree at the level of tropical cycles.

4.1. **q-deformation.** Many of the objects we have encountered so far admit natural "quantizations", i.e. q-deformations. A general discussion of q-deformation in the present context may be found in [FS2].

Kontsevich and Soibelman [KS] deform a general $\mathfrak{g}_\Gamma$ to an *associative, noncommutative* algebra $\mathfrak{g}_q$ over $\mathbb{C}[q^{\pm\frac{1}{2}}]$, generated by $\hat{e}_\gamma, \gamma \in \Gamma$. The classical product is quantized to

$$\hat{e}_\alpha \hat{e}_\beta = q^{\frac{1}{2}\langle\alpha,\beta\rangle} \hat{e}_{\alpha+\beta}.$$

In the quantization the Lie bracket is the natural one given by the commutator. In other words we are now thinking of the $\hat{e}_\gamma$ as *operators* (as opposed to the classical bracket, which corresponds to a Poisson bracket of the e_γ seen as *functions*). Namely we set

$$[\hat{e}_\alpha, \hat{e}_\beta] = (q^{\frac{1}{2}\langle\alpha,\beta\rangle} - q^{-\frac{1}{2}\langle\alpha,\beta\rangle})\hat{e}_{\alpha+\beta}.$$

Since this is the commutator bracket of an associative algebra, $\mathfrak{g}_q$ is automatically Poisson. Its completion $\widehat{\mathfrak{g}}_q$ is constructed precisely as in the case of $\widehat{\mathfrak{g}}_\Gamma$. The Poisson automorphisms T^Ω_β of (2.8) also admit natural quantizations given by the algebra automorphisms of $\widehat{\mathfrak{g}}_q$ acting by

$$\mathbf{U}^\Omega((-q^{\frac{1}{2}})^n \hat{e}_\alpha)(\hat{e}_\beta) = \hat{e}_\beta \prod_{j=0}^{\langle\alpha,\beta\rangle-1} \left(1 + (-1)^n q^{j+\frac{n+1}{2}} \hat{e}_\alpha\right)^\Omega \prod_{j=\langle\alpha,\beta\rangle}^{-1} \left(1 + (-1)^n q^{j+\frac{n+1}{2}} \hat{e}_\alpha\right)^{-\Omega}$$

(notice that the precise quantization depends on an additional shift parameter $n \in \mathbb{Z}$). A continuous family of stability data on $\mathfrak{g}_q$ is again given by a function $\Omega(\alpha, Z)$ such that the slope-ordered product $\prod^{\rightarrow, Z} \mathbf{U}^{\Omega_n(\alpha,Z)}((-q^{\frac{1}{2}})^n \hat{e}_\alpha)$ is locally constant.

The main enumerative objects we considered, namely the tropical invariants $N^{\rm trop}(\mathbf{w})$, also have a natural quantization $\widehat{N}^{\rm trop}(\mathbf{w})$ discovered by Block and Göttsche [BG]. In [FS2] we adapted the tropical vertex technique to this deformed setting.

On the other hand the choice of a quantization of the operator $\mathcal{Z}$ of (3.2) appears to be less standard. In the predecessor [FS1] to [FGS] we proposed to q-deform $\mathcal{Z}$ to an operator $\mathcal{Z}_q$ acting on suitable $\mathrm{GL}(\widehat{\mathfrak{g}}_q)$-valued holomorphic functions. Consider integral transforms $\Lambda_{\alpha,n,j}$ acting on suitable $\widehat{\mathfrak{g}}_q$-valued holomorphic functions $\phi(z)$ by

$$\Lambda_{\alpha,n,j}[\phi](z) = \int_{\ell_\alpha(Z)} \frac{dw}{w}\frac{w+z}{w-z} \log\left(1 + q^{j+\frac{n+1}{2}}\phi(w)\right).$$

Define $\widehat{X}^0(z)$ with values in $\mathrm{GL}(\widehat{\mathfrak{g}}_q)$ by

$$\widehat{X}^0(z)(\widehat{e}_\alpha) = \widehat{e}_\alpha \exp(z^{-1}Z(\alpha) + z\bar{Z}(\alpha)).$$

Let now $\widehat{\phi}(z)$ denote a $\mathrm{GL}(\widehat{\mathfrak{g}}_q)$-valued holomorphic function. Under the usual integrability conditions, we can define a new $\mathrm{GL}(\widehat{\mathfrak{g}}_q)$-valued holomorphic function by

$$\mathcal{Z}_q[\widehat{\phi}](z)(\widehat{e}_\alpha) = \widehat{X}^0(z)(e_\alpha)\prod_{\gamma}^{z} \exp\left(\sum_n \Omega_n(\gamma, Z)\left(\sum_{j=0}^{\langle\gamma,\alpha\rangle-1} - \sum_{j=\langle\gamma,\alpha\rangle}^{-1}\right)\Lambda_{\gamma,n,j}[\widehat{\phi}(e_\gamma)](z)\right), \tag{4.2}$$

where the operator $\prod^z_\gamma$ writes the ensuing factors from left to right according to the clockwise order of $Z(\gamma)$, starting from the basepoint $z \in \mathbb{C}^*$. A similar proposal has been made recently on physical grounds by Cecotti, Neitzke and Vafa [CNV] section 4.2.2. Our $\prod^z_\gamma$ seems to be closely related to their time ordering operator (where they regard the $\mathbb{P}^1$ variable z as "time"). The analysis of the q-TBA equation

$$\mathcal{Z}_q[\widehat{\phi}] = \widehat{\phi} \tag{4.3}$$

seems to be rather more complicated than the case of (3.2) even in our current purely algebraic setting.

However in the special case of $\mathcal{C}^+ \subset \mathrm{Stab}(Q_\kappa)$ treated above the operator (4.2) computed along the rays generated by $[S_i]$, $i = 1, 2$ simplifies dramatically to

$$\mathcal{Z}_q[\widehat{\phi}](z)(\widehat{e}_{r[S_i]}) = \widehat{X}^0(\widehat{e}_{r[S_i]}) \exp\left(\left(\sum_{k=0}^{\langle[S_j],r[S_i]\rangle-1} - \sum_{k=\langle[S_j],r[S_i]\rangle}^{-1}\right)\Lambda_{\gamma,1,j}[\widehat{\phi}(e_{[S_2]})](z)\right) \tag{4.4}$$

and the equation (4.3) can be solved iteratively from $X^0(z)$ precisely as (3.3). The solution is expressed in terms of q-deformed weight functions $G_{T,q}(z, Z)$ and q-deformed rational weights $W_{T,q}(Z) \in \mathbb{Q}[q^{\pm\frac{1}{2}}]$. The analogues of Theorems 4.1 and 4.3 also hold, to wit, the $G_{T,q}(z, Z)$ admit an expansion just like (4.1), with $\varepsilon(C_i)$

replaced by uniquely determined $\varepsilon_q(C_i) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$, and suitable weighted sums of contributions $\varepsilon_q(C_i)$ yield Block-Göttsche invariants. Explicit formulae are given in [FS1] section 4. Indeed this analysis of the equation (4.4) was the original motivation for our study of Block-Göttsche invariants through wall-crossing identities [FS2].

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